

Young's Experiment, Schrödinger's Spread and Spontaneous Intrinsic Decoherence

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The two slits interference pattern for single particle is revisited, showing its strict relation to the free particle Schrödinger's spread and describing the optical analogy. We explicitly show the possibility that spontaneous intrinsic decoherence (SID) can destroy interference pattern and that decoherence becomes stronger at the macroscopic limit.

Key words: Quantum Mechanics; Interference; Decoherence.

1. Young's Experiment: Two Slits Interference

Let us consider a beam of particles moving along the z axis with the velocity \bar{V} (see Fig. 1) and assume that the beam intensity is so low that one particle at a time arrives at the screen F , where there are two slits. The distance between the slits is $2d$, and 2σ is their width. When the particle passes through the slits ($t = 0$), its wave function (along the \hat{x} axis) is given by

$$\Psi_0(x) = c(\Psi_1(x) + \Psi_2(x)), \quad (1)$$

where c is a normalization constant and

$$\Psi_j(x) = \exp \left\{ -\frac{(x - x_j)^2}{4\sigma^2} \right\}, \quad j = 1, 2. \quad (2)$$

Here $|\Psi_j(x)|^2$ are minimum uncertainty wave packets centered at x_j , with zero average momentum and width σ . We take $x_1 = -x_2 = d$ and we assume $d \gg \sigma$, so that the two wave packets do not overlap and the interference term is negligible. The time evolution of $\Psi_j(x)$ can be written as [1]

$$\Psi_j(x, t) = c(t) \exp \left\{ -\frac{(x - x_j)^2}{4\sigma(t)^2} \left(1 - i \frac{\sigma_v}{\sigma} t \right) \right\}, \quad (3)$$

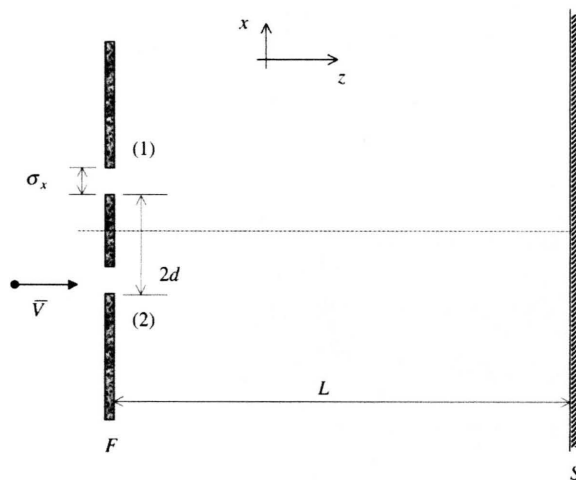


Fig. 1. Setup of the Young's experiment described in the text.

where $\sigma_v = \hbar/2m\sigma$ from Heisenberg's uncertainty principle, and

$$\sigma(t) = \sqrt{\sigma^2 + \sigma_v^2 t^2} = \sqrt{\sigma^2 + \left(\frac{\hbar}{2m\sigma} \right)^2 t^2} \quad (4)$$

is the well-known time dependent Schrödinger spread.

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From now on we make the change of parameters

$$\bar{x} = \frac{x}{\sigma}, \quad \bar{d} = \frac{d}{\sigma}, \quad \bar{t} = \frac{\sigma_v t}{\sigma} = \frac{\hbar}{2m\sigma^2} t \quad (5)$$

and give ω the unwonted definition (8).

The advantage of this scaling is that the interference pattern will depend only on the parameter \bar{d} , moreover the dimensionless time \bar{t} represents the ratio between the Schrödinger-spread at time t and the starting spread σ . From (1), (3) and (5) we see that the probability density $P(\bar{x}, \bar{t}) = |\Psi_1(\bar{x}, \bar{t}) + \Psi_2(\bar{x}, \bar{t})|^2$ of finding a particle in \bar{x} at time \bar{t} is

$$P(\bar{x}, \bar{t}) = c(\bar{t}) (G_+ + G_- + 2\sqrt{G_+ G_-} \cos \omega \bar{t}), \quad (6)$$

where

$$c(\bar{t}) = \frac{1}{2\sqrt{2\pi}\sqrt{1+\bar{t}^2}} \frac{1}{(1+e^{-\bar{d}^2/2})}, \quad (7)$$

$$G_{\pm} = \exp \left\{ -\frac{(\bar{x} \pm \bar{d})^2}{2(1+\bar{t}^2)} \right\}, \quad \omega = \frac{\bar{x}\bar{d}}{1+\bar{t}^2}. \quad (8)$$

Note that the time dependent interference term $\cos \omega \bar{t}$ is modulated by

$$\sqrt{G_+ G_-} = \exp \left\{ -\frac{\bar{x}^2}{2(1+\bar{t}^2)} \right\} \exp \left\{ -\frac{\bar{d}^2}{2(1+\bar{t}^2)} \right\}. \quad (9)$$

Equation (9), under the condition

$$\bar{t} \gg \bar{d} \gg 1, \quad (10)$$

can be approximated by

$$\sqrt{G_+ G_-} \approx \exp \left\{ -\frac{\bar{x}^2}{2\bar{t}^2} \right\}, \quad (11)$$

i. e. one has a Gaussian modulation. Hence the interference term is relevant only for $\bar{x} \ll \bar{t}$ ($x \ll \sigma_v t$). We underline that condition (10) is equivalent to

$$\sigma_v t \gg d \gg \sigma, \quad (12)$$

which implies complete overlap of the wave functions. Therefore, the interference pattern provides a direct evidence of Schrödinger's spread. The interference pattern evolution and the relevance of the conditions given above are shown in Figure 2.

Now, the visibility, as usually defined, is given by $F = \text{sech}(\omega)$, with $\omega \approx \bar{x}\bar{d}/\bar{t}^2$. Hence the condition to have good visibility ($F \approx 1$) is $\omega \ll 1$, which is certainly satisfied when $\bar{x}/\bar{t} \ll 1$ and $\bar{d}/\bar{t} \ll 1$, in agreement with (10).

Even if the relation between (6 - 9) and Young's interference experiment of optics appears obscure, it will be clarified later, transforming properly the time dependence into a space dependence.

The interference pattern maxima occur at $\omega \bar{t} = 2n\pi$ (see (6)), so that by (8) and (5) we can write (in the limit of (10))

$$\bar{x}_n = \frac{2n\pi\bar{t}}{\bar{d}}. \quad (13)$$

Hence, from (11), these maxima are modulated by

$$\exp \left\{ -\frac{\bar{x}_n^2}{2\bar{t}^2} \right\} = \exp \left\{ -\frac{1}{2} \left(\frac{2n\pi}{\bar{d}} \right)^2 \right\}, \quad (14)$$

so that the number of visible fringes is proportional to $\bar{d} = d/\sigma$, as in the optical case.

The optical analogy with the Young interference experiment can be obtained interpreting $P(x, t)$ (obtained from (6) using (5)) as the probability density of finding a particle on a screen S at distance $L \gg d$, such that $t = L/\bar{V}$, where \bar{V} is the velocity perpendicular to the plane \hat{x} of the slits (see Fig. 1). In this way (13) becomes

$$\frac{x_n}{L} = \frac{\lambda}{2d} n, \quad \text{where } \lambda = \frac{h}{m\bar{V}} = 2\pi\lambda, \quad (15)$$

which represents the usual condition for constructive interference.

All the previous equations can be translated into optical terms with the identification $t = L/\bar{V}$ and $\lambda = h/m\bar{V}$; for example, the condition $\bar{t} \gg 1$ becomes

$$\frac{L}{R} \gg 1, \quad (16)$$

where $R = 2\sigma^2/\lambda$ is the well known Rayleigh range, which is a measure of the divergence of a Gaussian beam whose transverse width is σ . The conditions $\bar{d} \ll \bar{t}$ and $\bar{x} \ll \bar{t}$ become

$$\frac{d}{L} \ll \vartheta_{\text{diff}}, \quad \vartheta = \frac{x}{L} \ll \vartheta_{\text{diff}}, \quad \vartheta_{\text{diff}} = \frac{\lambda}{2\sigma}, \quad (17)$$

i. e. the angle d/L and the observation angle $\vartheta = x/L$ must be smaller than the diffraction angle ϑ_{diff} .

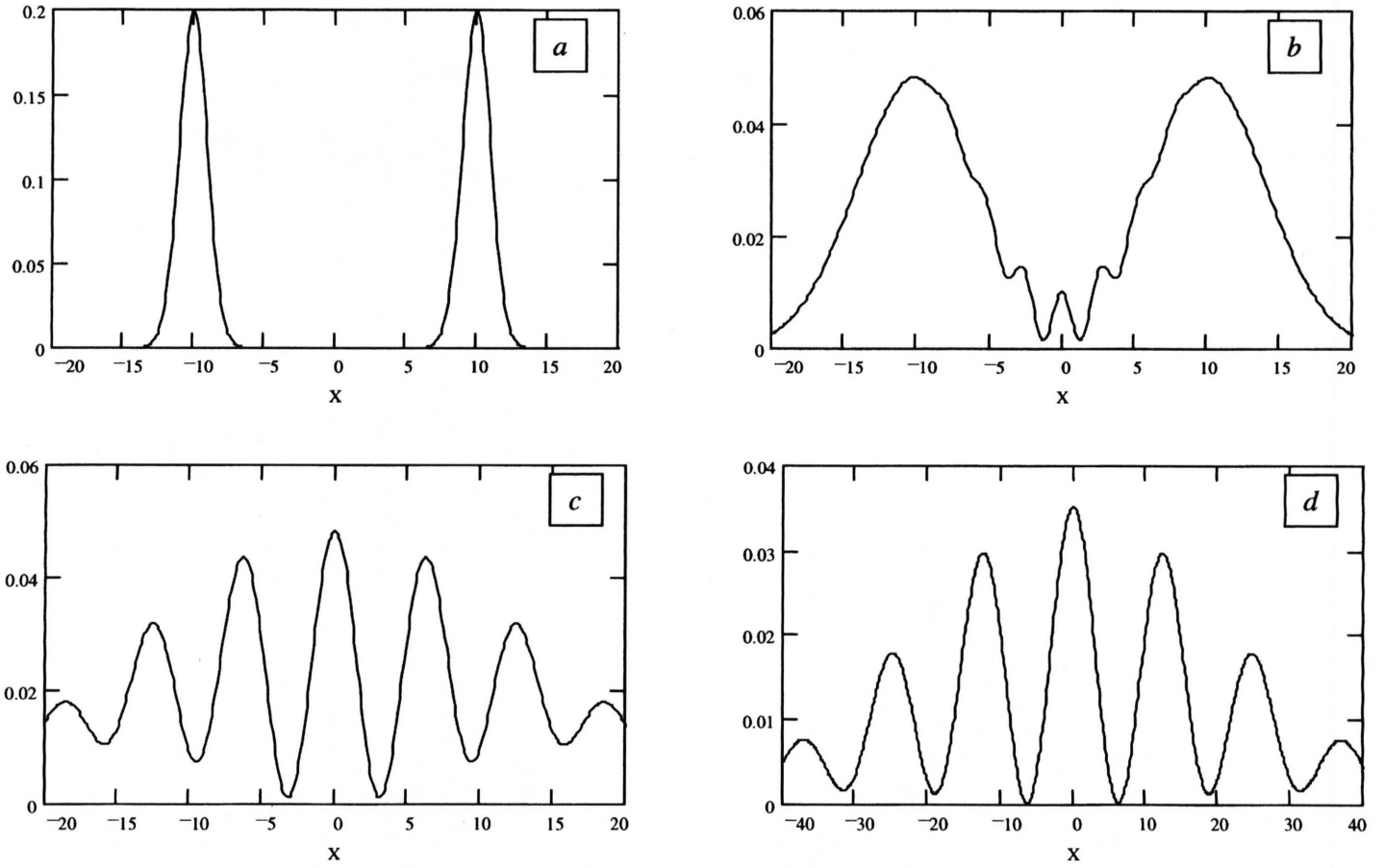


Fig. 2. Time evolution of the interference pattern with $d/\sigma = 10$ for different values of the dimensionless time \bar{t} : a) $\bar{t} = 0$; b) $\bar{t} = 4$; c) $\bar{t} = 10$; d) $\bar{t} = 20$.

The first of conditions (17) guarantees that particles arrives at the center of the screen S (at $x = 0$). The translation into optical terms can be similarly done for the other equations. For example (11), using the previous definitions, can be written under the form of the well-known diffraction envelope:

$$\sqrt{G_+ G_-} \approx \exp \left\{ -\frac{x^2 (2\sigma)^2}{2L^2 \lambda^2} \right\} = \exp \left\{ -\frac{\vartheta^2}{2\vartheta_{\text{diff}}^2} \right\}. \quad (18)$$

2. Decoherence and Spontaneous Intrinsic Decoherence (SID)

First of all we recall the main points of the formalism used here to describe *non dissipative decoherence* and, in particular, *spontaneous intrinsic decoherence* (SID) of the interference pattern, i. e. a decoherence not due to interaction with the environment or fluctuation of some internal parameter, but of purely quantum mechanical origin (see also [2, 3]).

Let us consider an initial state described by the density operator $\rho(0)$ and consider the case of a random evolution time. The experimentally observed state is not described by the usual density matrix of the whole system $\rho(t)$, but by its time averaged counterpart [2]

$$\bar{\rho}(t) = \int_0^\infty dt' \mathcal{P}(t, t', \tau) \rho(t'), \quad (19)$$

where $\rho(t') = \exp\{-i\mathcal{L}t'\}\rho(0)$ is the usual unitarily evolved density operator from the initial state and $\mathcal{L} \dots = [H, \dots]/\hbar$. Hence one can write

$$\bar{\rho}(t) = V(t)\rho(0), \quad (20)$$

where

$$V(t) = \int_0^\infty dt' \mathcal{P}(t, t', \tau) e^{-i\mathcal{L}t'}. \quad (21)$$

In [2], the function $\mathcal{P}(t, t', \tau)$ has been determined so to satisfy the following conditions: i) $\bar{\rho}(t)$ must be a density operator, i. e. it must be self-adjoint, positive-definite, and with unit-trace. This leads to the condition that $\mathcal{P}(t, t', \tau)$ must be non-negative and normalized, i. e. a probability density in t' so that (19) is a completely positive mapping. ii) $V(t)$ satisfies the semigroup property $V(t_1 + t_2) = V(t_1)V(t_2)$, with $t_1, t_2 \geq 0$. These requirements are satisfied by [2]

$$V(t) = (1 + iL\tau)^{-t/\tau}, \quad (22)$$

$$\mathcal{P}(t, t', \tau) = \frac{e^{-t'/\tau}}{\tau} \frac{(t'/\tau)^{(t/\tau)-1}}{\Gamma(t/\tau)}, \quad (23)$$

and the mapping

$$\rho(t) \mapsto \bar{\rho}(t) = \int_0^\infty dt' \mathcal{P}(t, t', \tau) \rho(t') \quad (24)$$

is a *positive linear mapping*. The above expressions $V(t)$ and $\mathcal{P}(t, t', \tau)$ satisfy (21) accordingly to the Γ -function integral identity [4, 5]. $\mathcal{P}(t, t', \tau)$ is the well known positive definite Γ -distribution function for the random variable t' and it parametrically depends on the *clock time* t and on the scaling time τ . The parameter τ characterizes the strength of the evolution time fluctuations. When $\tau \rightarrow 0$, we have $\mathcal{P}(t, t', \tau) \rightarrow \delta(t - t')$ so that $\bar{\rho}(t) = \rho(t)$ and $V(t) = \exp\{-i\mathcal{L}t\}$ is the usual unitary evolution. However, for finite τ , the evolution operator $V(t)$ of (22) describes decoherence in the energy representation (i.e. the approach to diagonal form [2, 3]), whereas the diagonal matrix elements remain constant, i. e. the energy is still a constant of motion (*non dissipative decoherence*). Finally, we showed in [6] that the above formalism is also valid when τ is a function of t , obviously loosing the semigroup property for the time evolution.

In the case of interference patterns, in agreement with the mapping (24) and for what was said above, the observed probability distribution of finding a particle on the screen S must be a time average of (6), i. e.

$$\bar{P}(\bar{x}, \bar{t}) \equiv \langle \bar{x} | \bar{\rho}(\bar{t}) | \bar{x} \rangle = \int_0^\infty dt' \mathcal{P}(\bar{t}, \bar{t}', \bar{\tau}) P(\bar{x}, \bar{t}'). \quad (25)$$

$\bar{\tau}$ is a characteristic time of the system which rules the uncertainty in the evolution time, and we used the scaling (5). In this way $\bar{P}(\bar{x}, \bar{t})$ is ruled by the *adimensional* characteristic time

$$\bar{\tau} \equiv \frac{\sigma_v}{\sigma} \tau. \quad (26)$$

Now we return to the unscaled notation and we propose to consider τ as the uncertainty of the arrival time of particles on the screen. This arrival time

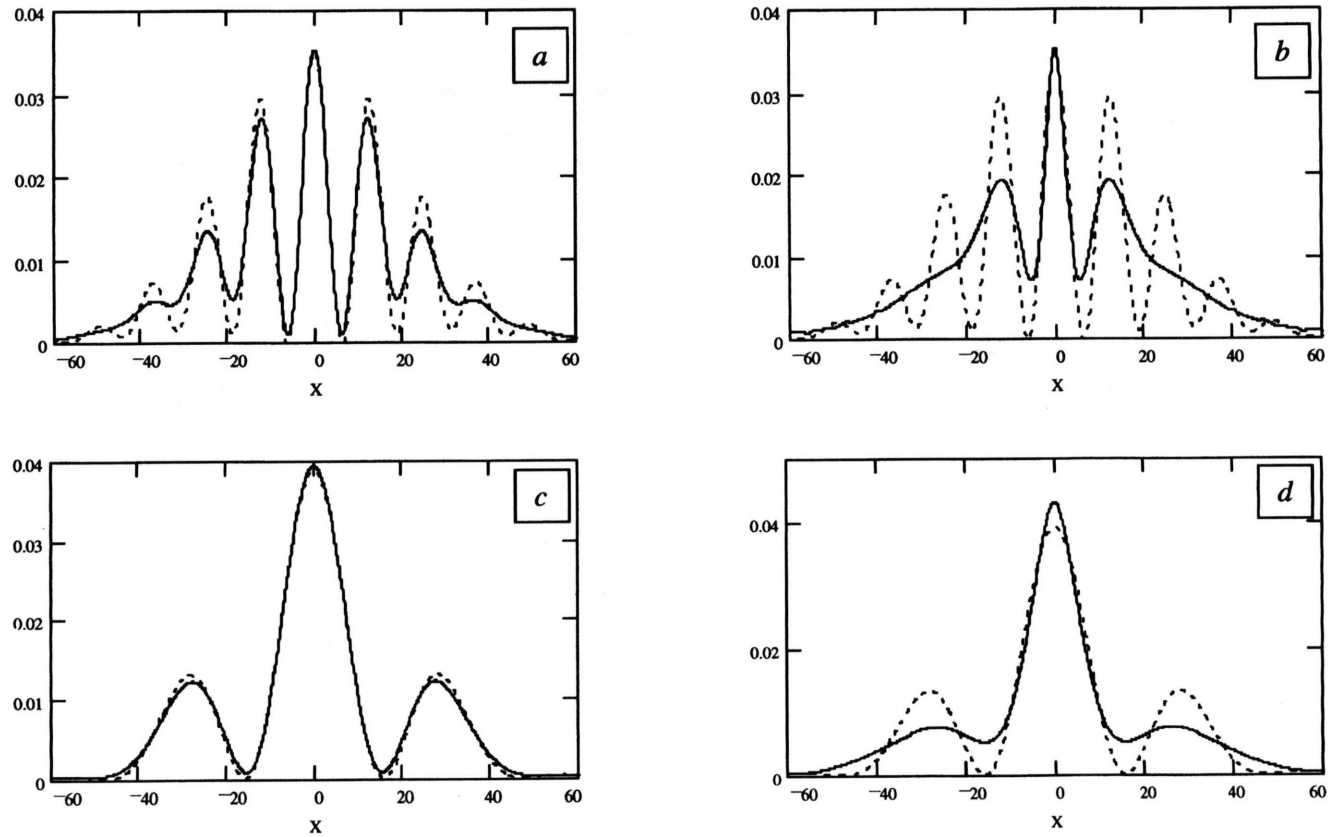


Fig. 3. Behaviour of interference pattern with decoherence (solid line) and without decoherence (dotted line) at time \bar{t} for different value of d/σ and a : a) $d/\sigma = 10$, $\bar{t} = 20$, $a = 0.01$; b) $d/\sigma = 10$, $\bar{t} = 20$, $a = 0.1$; c) $d/\sigma = 4$, $\bar{t} = 20$, $a = 0.01$; d) $d/\sigma = 4$, $\bar{t} = 20$, $a = 0.1$.

is defined as $t = L/\bar{v}$ (here \bar{v} is the *mean* particle velocity), so that [6]

$$\tau = \frac{\Delta_z(t)}{\bar{v}}, \text{ where } \Delta_z(t) = \sqrt{\sigma_z^2 + \sigma_{\bar{v}}^2 t^2 + \Delta_{\bar{v}}^2 t^2}. \quad (27)$$

Note that τ is always greater than $\tau_{\min} = \sigma_z/\bar{v} = \hbar/2\sigma(\mathcal{H}_z)$, where \mathcal{H}_z is the kinetic energy along the z axis and $\sigma(\mathcal{H}_z)$ its standard deviation, in agreement with the Tam-Mandelstam inequality, i. e. the time-energy uncertainty relation [7]. The expression (27) is obtained by summing up the square of two errors: an intrinsic one, $\sigma_z^2 + \sigma_{\bar{v}}^2 t^2$ (with $\sigma_z \sigma_{\bar{v}} = \hbar/2$), i. e. the quantum Schrödinger free particle spread along the z axis, and an extrinsic one, $\Delta_{\bar{v}}^2 t^2$, i. e. the spread due to the classical uncertainty of the velocity distribution. A more rigorous derivation of (27) is given in [6]. Assuming t large enough, so that $(\sigma_{\bar{v}}^2 + \Delta_{\bar{v}}^2)t^2 \gg \sigma_z^2$, from (26) and (27) one directly obtains

$$\bar{\tau} = a\bar{t} \quad \text{with} \quad a = \frac{\sqrt{\sigma_{\bar{v}}^2 + \Delta_{\bar{v}}^2}}{\bar{v}}. \quad (28)$$

If we assume $\bar{\tau}/\bar{t} = \tau/t = a \ll 1$, $\mathcal{P}(\bar{t}, \bar{t}', \bar{\tau})$ can be approximated with a Gaussian, so that we can perform the average (25), obtaining the analytical solution

$$\bar{P}(\bar{x}, \bar{t}) = b(\bar{t}) \{ G_+ + G_- + 2\sqrt{G_+ G_-} e^{-D} \cos \omega \bar{t} \}, \quad (29)$$

where $b(\bar{t})$ is a normalization constant and

$$D = \frac{\bar{x}^2 \bar{d}^2}{2} \frac{\bar{\tau}}{\bar{t}^3} = \frac{x^2 (2d)^2}{2L^2 \lambda^2} a, \quad (30)$$

where we used (5) and (28). The behaviour of $\bar{P}(\bar{x}, \bar{t})$ is shown in Fig. 3 for different values of d/σ and a . The term e^{-D} is another Gaussian *decoherence envelope* not due to diffraction, as the one of (18), but due to velocity spread. It is very important and significative to note that it goes as $(2d)^2$, i. e. the square of the distance between slits. In general, the diffraction envelope and the decoherence one are both present in experiments. The latter becomes visible as soon as $D \gtrsim 1$, and it dominates the diffraction when

$$a > \left(\frac{\sigma}{d} \right)^2 \quad (31)$$

which is the main condition to observe decoherence and it puts a lower limit to the relative velocity spread a .

We can also write (30) as

$$D = \frac{1}{2} \left(\frac{\vartheta}{\vartheta_{\text{int}}} \right)^2 a, \quad (32)$$

where $\vartheta_{\text{int}} = \lambda/2d$ (see (15)) and $\vartheta \approx x/L$. If one evaluates D in interference pattern maxima (15), one obtains

$$D = 2\pi^2 n^2 a. \quad (33)$$

Hence decoherence becomes more and more efficient as n and a increase (see Fig. 3).

Now, if we assume (28) and if we focus our attention on the limit $\Delta_{\bar{v}}^2 \gg \sigma_{\bar{v}}^2$ (*classical limit*), then (32) is only ruled by the ratio $a = \Delta_{\bar{v}}/\bar{v}$, so that decoherence has a classical origin. On the other hand, if we have $\Delta_{\bar{v}}^2 \ll \sigma_{\bar{v}}^2$ (*quantum limit*), we obtain the *spontaneous intrinsic decoherence* (SID), due to the quantum velocity spread $\sigma_{\bar{v}} = \hbar/2m\sigma_z$, and so $a = \sigma_{\bar{v}}/\bar{v}$. So this decoherence has an intrinsic, Quantum Mechanical origin related to Heisenberg's uncertainty principle.

Finally, at the macroscopic limit λ becomes very small. However, if one also decreases σ , so that λ/σ remains constant, hence the diffraction envelope does not suppress the interference pattern, on the contrary the decoherence envelope kills the interference keeping constant the distance between slits (see (30)). So, in our opinion, the correct statement for the classical limit in particle interference experiment is $\lambda/d \rightarrow 0$.

3. Conclusions

We showed that the interference pattern of a two slits experiment is strictly connected to the free particle Schrödinger spread. Moreover, fringes visibility is reduced by a diffraction envelope and by a decoherence envelope. The latter is due to the formalism of [2], where the randomness of evolution time is due to particle velocity spread. The spread *can* have a classical contribution (like thermal spread), and it *always* has a quantum contribution due to the Heisenberg's uncertainty principle. We specified the conditions under which the last is dominant, in such a case one should observe *spontaneous intrinsic decoherence*, i. e. not due to interaction with environment or to experimental fluctuation of parameters.

One can think that our results can be easily reproduced just performing an average with the classical velocity distribution along the \hat{z} axis. However this will give an expression of the decoherence exponent

D similar to (32), but with two basic differences: a^2 instead of a and a will be only a function of the classical contribution $\Delta_{\overline{\gamma}}$. From a quantitative view point, if $a \ll 1$ then one has a much smaller decoherence than the one of our approach.

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